The traffic statics problem in a road network

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Abstract

In this study we define and solve the traffic statics problem in an open diverge-merge network based on a multi-commodity kinematic wave model, whose entropy conditions are given by invariant flux functions derived from macroscopic merging and diverging rules. In this problem, we are interested in finding stationary states on all links when origin demands, destination supplies, and route choice proportions are constant. After discussing the properties of four types of stationary states on a road link and presenting stationary entropy conditions at both the merging and diverging junctions, we derive a system of algebraic equations as necessary conditions for all 16 combinations of stationary states on the two intermediate links. Under different network conditions in road capacities, route choice proportions, and merging priorities, we analytically show that the traffic statics problem always admits stationary solutions, which, however, may not be unique. In particular, such stationary solutions exist even under network conditions when an initially empty diverge-merge network can settle in persistent periodic oscillations after a long time. In the future, we will be interested in discussing the stability property of stationary states and studying the traffic statics problem in general networks. Analytical insights from the simpler traffic statics problems would be helpful for understanding complex traffic dynamics in a road network.

Keywords: Traffic statics problem; Diverge-merge network; Multi-commodity kinematic wave model; Merging and diverging rules; Boundary and initial conditions; Stationary states

1 Introduction

In order to better design, manage, and control a road network, it is critical to understand the underlying traffic dynamics, which are driven by people's choice behaviors in trips,

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destinations, modes, departure times, routes, lanes, and speeds, and constrained by network geometry and traffic control measures. Generally, traffic flow theories are employed to study traffic dynamics caused by interplays between drivers' car-following, lane-changing, merging, and diverging behaviors and various network conditions. Microscopic models have been developed to describe individual driver-vehicle units' movements (Pipes, 1953; Gazis et al., 1961; Newell, 2002; Gipps, 1986; Hidas, 2005). Macroscopic models, e.g., the Lighthill-Whitham-Richards (LWR) model (Lighthill and Whitham, 1955; Richards, 1956), gas kinetic (Prigogine and Andrews, 1960), and hydrodynamic models (Payne, 1971; Whitham, 1974), have been developed to describe the evolution of traffic densities and travel speeds at different locations on a road link.

Conceptually, traffic dynamics in a road network are determined by network conditions, initial conditions, boundary conditions[1], and traffic flow models. In the literature, traffic flow models for road links have been successfully used to explain such phenomena as shock and rarefaction waves (Lighthill and Whitham, 1955), spontaneous clusters (Kerner and Konhäuser, 1994), stability of platoons (Herman and Rothery, 1965), and so on. Recently, with the development of Cell Transmission Model (CTM) (Daganzo, 1995; Lebacque, 1996), there has been much interest in understanding traffic dynamics caused by such bottlenecks as merging and diverging junctions in an oversaturated network (e.g. Daganzo, 1999). In (Daganzo, 1996), a beltway network was shown to asymptotically converge to a complete gridlock state with certain constant demand patterns and merging priorities. In (Daganzo et al., 2011), a double-ring network was shown to have multiple stationary states, and bifurcations in the occurrence of these stationary states can be caused by different loading patterns. In (Jin, 2003, Section 7.3), by simulations with a commodity-based CTM with fair merging and first-in-first-out (FIFO) diverging rules, it was shown that periodic oscillations can develop in an initially empty diverge-merge network with two intermediate links, hereafter referred to as the DM2 network, shown in Figure 1. After a long time, these oscillations can damp or persist, even with constant origin demands, destination supplies, and route choice proportions. In (Jin and Zhang, 2005), such oscillations were replicated with Paramics, a microscopic traffic simulator, and it suggests that such periodic oscillations are caused by interactions among the diverging and merging junctions, not by the specific traffic flow models. In (Jin, 2009), it was shown that damped and persistent periodic oscillatory dynamics in the DM2 network are associated with circular propagation of shock and rarefaction waves on two intermediate links, and the diverging and merging junctions serve as switches for these waves.

The aforementioned and many other studies have suggested that complicated traffic dynamics can arise in a road network due to interactions among traffic streams at merging and diverging junctions. When dynamics in various physical systems are too complicated to analyze, one would usually first try to solve the simpler statics problem. In the same spirit, in order to achieve a systematic understanding of complex traffic dynamics in a road network,

[1]Boundary conditions include traffic signals and time-dependent travel demands on all routes, which can be either observed with detectors or determined by a traffic assignment process.

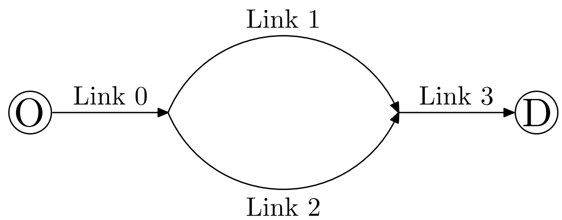

Figure 1: A diverge-merge (DM2) network with two intermediate links

we can first study the traffic statics problem, in which we are interested in finding stationary states on all links when all network and boundary conditions are time-independent. That is, if all links are in stationary states initially, the traffic patterns in the network remain the same along the time. The traffic statics problem can be defined for open networks, when boundary conditions in origin demands, destination supplies, and route choice proportions are all constant,[2] for closed networks, when we have periodic boundary conditions, or for mixed open-closed networks, when boundary conditions are either constant or periodic.

In the literature, there have been some studies related the traffic statics problem. For examples, (Daganzo, 1996) studied the traffic statics problem in an open beltway network, which has a gridlock solution; (Jin et al., 2009) defined stationary states on a link and studied the traffic statics problem on a closed inhomogeneous ring road, which has a unique stationary solution; and (Daganzo et al., 2011) studied the traffic statics problem in a closed double-ring network. In addition, the traffic statics problem in an open network is highly related to network loading problems with constant loading patterns. Traditionally, network loading problems have been studied with link performance functions (e.g. Wu et al., 1998) or with microscopic or mesoscopic simulation models (e.g. Barcelo and Casas, 2005; Lo, 1999). However, there exist no systematic definitions or solutions of such problems.

In this study, we attempt to define and solve the traffic statics problem in the open DM2 network within the framework of a network kinematic wave model (Jin, 2010b,a), whose entropy conditions are given by invariant flux functions derived from macroscopic merging and diverging rules. The kinematic wave model can be viewed as an infinitely-dimensional dynamical system. Thus solving the traffic statics problem is equivalent to finding its equilibrium state under constant boundary conditions. [3] Through this study, we attempt to make the following theoretical contributions:

- First, we rigorously define the traffic statics problem in the open DM2 network based

[2]For open networks, we can also define the traffic statics problem with constant origin demands, destination supplies, and turning proportions at all junctions. In addition, when vehicles have no predefined routes, the traffic statics problem can be defined with time-independent origin demands and destination supplies.

[3]In transportation systems, many other equilibrium states have been studied. For examples, the speed-density relation in the fundamental diagram consists of equilibrium states of car-following dynamics, and user equilibrium defined by (Wardrop, 1952) can be considered as an equilibrium point of route choice dynamics (Smith, 1984; Jin, 2007).

on network kinematic wave models. Such a definition can be extended for general road networks.

- Second, we solve stationary states in the DM2 network under different network conditions. As we know, on a road link with constant demand patterns, traffic becomes stationary after a long time under any initial conditions [4]. But (Jin, 2009) and related studies showed that this is not the case for the DM2 network: even under constant boundary conditions and constant network conditions, an initially empty DM2 network could settle in persistent periodic oscillations. Thus it suggests that the traffic statics problem may not admit a solution in a road network. In this study, we demonstrate the opposite analytically: the traffic statics problem always admits stationary solutions in the DM2 network.

- Third, we attempt to develop a systematic method for solving the traffic statics problem in the DM2 and other small networks.

Even though the DM2 network shown in Figure 1 looks to be simple with only one O-D pair, it contains two routes and the two most important types of network bottlenecks: a diverge and a merge. The DM2 network is believed to be the simplest network with a diverge, a merge, and more than one routes and, therefore, an ideal model network for studying traffic dynamics caused by network bottlenecks as well as route choice behaviors. [5] In addition, under appropriate conditions, many real road networks could be modeled by the DM2 network:

- For a freeway section with an adjacent frontage road, as shown in Figure 2, if a significant number of vehicles choose to exit the freeway from the upstream off-ramp, travel on the frontage road, and then enter the freeway through the downstream on-ramp, this network becomes effectively a DM2 network.

- For a grid network of unidirectional roads shown in Figure 3, a DM2 network is embedded (shown by the thicker lines), if the number of vehicles from O to D is significantly higher than those of other O-D pairs.

In this sense, studies on the DM2 network could be practically relevant too.

The rest of the paper is organized as follows. In Section 2, we present a kinematic wave model of traffic dynamics in the DM2 network. In Section 3, we define the traffic statics problem in the DM2 network and study properties of stationary states. In Section 4, we present stationary solutions under different network conditions and route choice proportions with a priority-based merging rule. In Section 5, we discuss stationary states for two special merging rules. In Section 6, we summarize our findings and present some discussions.

[4] The existence of asymptotic stationary solutions of scalar hyperbolic conservation laws was discussed in (Lax, 1972, Section 4).

[5] Some preliminary studies on route choice behaviors in a DM2 network can be found in (Jin, 2003, Section 7.2).

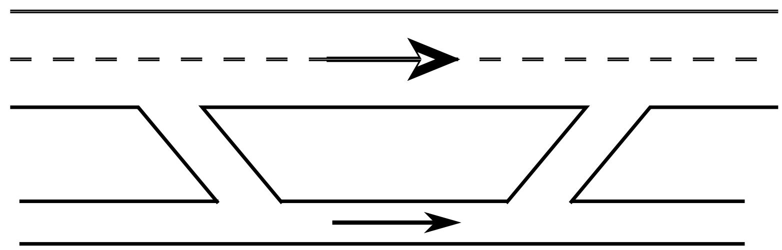

Figure 2: A freeway section with an adjacent frontage road

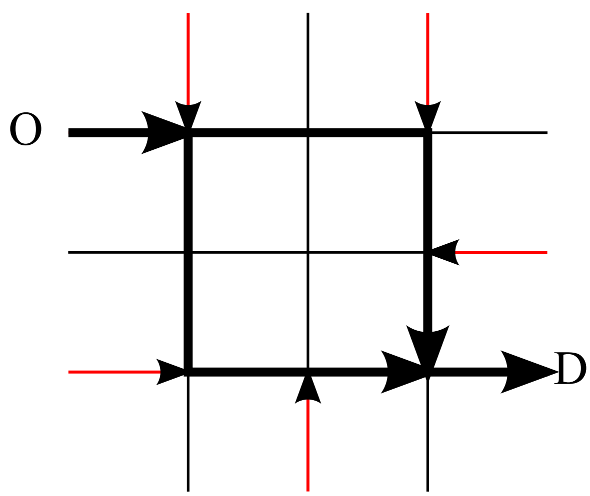

Figure 3: A three by three unidirectional grid network

2 A multi-commodity network kinematic wave model

For the DM2 network in Figure 1, we introduce two dummy links at the origin and destination and label them as links r and w, respectively. Then the set of links in the network is $A = \{r, 0, 1, 2, 3, w\}$. Route 1 uses links $r, 0, 1, 3, w$, and route 2 uses links $r, 0, 2, 3, w$. We consider all vehicles using route 1 as commodity 1, and those using route 2 as commodity 2. The network has four junctions: junction 0 connecting links r and 0, junction 1 connecting links 0, 1, and 2, junction 2 connecting links 1, 2, and 3, and junction 3 connecting links 3 and w. Here junctions 0 and 3 are linear, junction 1 is a diverge, and junction 2 is a merge.

2.1 Definitions of variables

For link $a \in A$, we introduce a link coordinate $x_a \in [0, X_a]$, where X_a is its length, and the positive direction of x_a is the same as traffic direction. For dummy links r and w, their lengths are zero. Then a point $x_a \in [0, X_a]$ on link a can be represented by a link coordinate (a, x_a). Any point inside a link has a unique coordinate, but junctions have multiple coordinates. For example, the coordinate of the diverging junction point can be $(1, X_1)$, $(2, 0)$, or $(3, 0)$. In this case, we denote $(1, X_1) \sim (2, 0) \sim (3, 0)$.

At a point (a, x_a) in the link-based coordinate system and time t, we define the following quantities as functions of (x_a, t): total density $k_a(x_a, t)$, speed $v_a(x_a, t)$, flow-rate $q_a(x_a, t)$, demand $d_a(x_a, t)$, and supply $s_a(x_a, t)$. In addition, on all links except links 1 and 2, we define $\xi_a(x_a, t)$ as the proportion of commodity 1 vehicles. For the purpose of notational simplicity, we will drop (x_a, t) from these variables unless necessary. Then the density and flow-rate of commodity 1 are $\xi_a k_a$ and $\xi_a q_a$, respectively.

At a point (a, x_a) ($a = 0, \cdots, 3$) and time t, we assume a speed-density relation $v_a = V_a(k_a)$ and flow-density relation $q_a = Q_a(k_a) = k_a V_a(k_a)$ (Greenshields, 1935). That is, all links in the DM2 network are homogeneous, and we do not consider impacts of special lanes (Daganzo, 1997) or heterogeneous traffic (Wong and Wong, 2002; Zhang and Jin, 2002). Generally, $q_a = Q_a(k_a)$ is unimodal and attains its capacity $C_a = Q_a(k_{a,c})$ at a critical density, $k_{a,c}$. Then traffic demand and supply at a point can be defined as (Engquist and Osher, 1980; Daganzo, 1995; Lebacque, 1996)

$$d_a = D_a(k_a) \equiv Q_a(\min\{k_{a,c}, k_a\}), \quad (1a)$$
$$s_a = S_a(k_a) \equiv Q_a(\max\{k_{a,c}, k_a\}), \quad (1b)$$

where $D_a(k_a)$ and $S_a(k_a)$ correspond to the increasing and decreasing branches of $Q_a(k_a)$, respectively. In the sense of (Daganzo, 1995), d_a and s_a are "the maximum flows that can be sent and received" by a location, respectively. From the definition of demand and supply, we can see that the demand-supply pair can uniquely determine density and flow-rate:

$$q_a = \min\{d_a, s_a\}, \quad (2a)$$
$$C_a = \max\{d_a, s_a\}, \quad (2b)$$

$$k_a \quad = \quad R_a(d_a/s_a) \equiv \begin{cases} D_a^{-1}(C_a d_a/s_a), & d_a \leq s_a \\ S_a^{-1}(C_a s_a/d_a), & d_a > s_a \end{cases} \tag{2c}$$

For example, for a triangular fundamental diagram (Munjal et al., 1971; Newell, 1993), $Q_a(k_a) = \min\{v_{a,f}k_a, \frac{n_a}{\tau_a}(1 - \frac{k_a}{k_{a,j}})\}$, where $v_{a,f}$ is the free flow speed, τ_a the time gap, n_a the number of lanes, and $k_{a,j}$ the jam density, we have

$$\begin{aligned} d_a &= \min\{v_{a,f}k_a, v_{a,f}k_{a,c}\}, \\ s_a &= \min\{v_{a,f}k_{a,c}, \frac{n_a}{\tau_a}(1 - \frac{k_a}{k_{a,j}})\}, \\ R_a(d_a/s_a) &= \begin{cases} \frac{C_a}{v_{a,f}} \frac{d_a}{s_a}, & d_a \leq s_a \\ (1 - \frac{C_a \tau_a}{n_a} \frac{s_a}{d_a}) k_{a,j}, & d_a > s_a \end{cases} \end{aligned}$$

where $k_{a,c} = \frac{n_a k_{a,j}}{n_a + \tau_a k_{a,j} v_{a,f}}$, and $C_a = v_{a,f}k_{a,c}$. For $d_a/s_a \in [0, \infty)$, $R_a(d_a/s_a)$ is a continuous and increasing function in d_a/s_a: $R_a(0) = 0$, $R_a(1) = k_{a,c}$, and $R_a(\infty) = k_{a,j}$.

2.2 A link-based kinematic wave model

On link a, traffic dynamics are described by the following LWR model (Lighthill and Whitham, 1955; Richards, 1956)

$$\frac{\partial k_a}{\partial t} + \frac{\partial k_a V_a(k_a)}{\partial x_a} \quad = \quad 0, \quad a = 0, \cdots, 3. \tag{3a}$$

To track dynamics of commodity 1 flows on links 0 and 3 we apply a commodity-based LWR model (Lebacque, 1996):

$$\frac{\partial \xi_a k_a}{\partial t} + \frac{\partial \xi_a k_a V_a(k_a)}{\partial x_a} \quad = \quad 0, \quad a = 0, 3 \tag{3b}$$

which is equivalent to $\frac{\partial \xi_a}{\partial t} + V_a(k_a)\frac{\partial \xi_a}{\partial x_a} = 0$.

For the DM2 network, there are totally six equations from (3): on links 1 and 2, traffic dynamics are described by the traditional LWR model (3a); but on links 0 and 3, both (3a) and (3b) have to be used to describe traffic dynamics for both commodities. For the traditional LWR model, a so-called entropy condition has to be applied to pick out unique, physical weak solutions (Lax, 1972; Ansorge, 1990). But the multi-commodity LWR model, (3), is a system of six non-strictly hyperbolic conservation laws, and traditional entropy conditions cannot be easily extended at the merging and diverging junctions.

In (Jin et al., 2009; Jin, 2010b,a), it was shown that the CTM flux functions in upstream demands, downstream supplies, and turning proportions can be used as entropy conditions, such that the Riemann problem can be uniquely solved at linear, merging, and diverging junctions. That is, the multi-commodity kinematic wave model in (3) is well defined with the following entropy conditions:

1. If a point (a, x_a) has only one upstream link and one downstream link, then the total and commodity 1 fluxes through this point are given by

$$q_a(x_a,t) = \min\{d_a(x_a^-,t), s_a(x_a^+,t)\}, \tag{4a}$$
$$\xi_a(x_a,t)q_a(x_a,t) = \xi_a(x_a^-,t)q_a(x_a,t), \tag{4b}$$

where x_a^- and x_a^+ are the upstream and downstream points of (a, x_a), respectively.

   (a) At the origin junction $(r,0) \sim (0,0)$, if the demand at origin, $d_r(0^-,t)$, and the proportion of commodity 1, $\xi_r(0^-,t)$, are given, the entropy condition (4) becomes

$$q_r(0,t) = q_0(0,t) = \min\{d_r(0^-,t), s_0(0^+,t)\}, \tag{5a}$$
$$\xi_r(0,t)q_r(0,t) = \xi_r(0^-,t)q_r(0,t). \tag{5b}$$

   (b) At the destination junction $(3, X_3) \sim (s,0)$, if the supply at the destination, $s_r(0^+,t)$, is given, the entropy condition (4) becomes

$$q_3(X_3,t) = q_w(0,t) = \min\{d_3(X_3^-,t), s_w(0^+,t)\}, \tag{6a}$$
$$\xi_w(0,t)q_w(0,t) = \xi_3(X_3^-,t)q_w(0,t). \tag{6b}$$

2. At the diverge $(0, X_0) \sim (1,0) \sim (2,0)$, boundary fluxes are given by

$$q_0(X_0,t) = \min\{d_0(X_0^-,t), \frac{s_1(0^+,t)}{\xi_0(X_0^-,t)}, \frac{s_2(0^+,t)}{1-\xi_0(X_0^-,t)}\}, \tag{7a}$$
$$q_1(0,t) = q_0(X_0,t)\xi_0(X_0^-,t), \tag{7b}$$
$$q_2(0,t) = q_0(X_0,t)(1-\xi_0(X_0^-,t)), \tag{7c}$$

which is the FIFO diverging model in (Daganzo, 1995).

3. At the merge $(1, X_1) \sim (2, X_2) \sim (3,0)$, boundary fluxes are given by

$$q_3(0,t) = \min\{d_1(X_1^-,t) + d_2(X_2^-,t), s_3(0^+,t)\}, \tag{8a}$$
$$q_1(X_1,t) = \min\{d_1(X_1^-,t), \max\{s_3(0^+,t) - d_2(X_2^-,t), \beta s_3(0^+,t)\}\}, \tag{8b}$$
$$q_2(X_2,t) = \min\{d_2(X_2^-,t), \max\{s_3(0^+,t) - d_1(X_1^-,t), (1-\beta)s_3(0^+,t)\}\}, \tag{8c}$$

which is the priority-based merging rule with a merging priority β (Daganzo, 1995). When $\beta = \frac{C_1}{C_1+C_2}$, this is the fair-merging rule (Jin, 2010b).

The multi-commodity kinematic wave model, (3), together with the entropy conditions above, can be considered an infinitely dimensional dynamical system, or a semigroup (Bressan, 1996), for which the state variables $k_a(x_a,t)$ $(a = 0, \cdots, 3)$ and $\xi_a(x_a,t)$ $(a = 0, 3)$ at any time can be uniquely solved from given initial conditions in $k_a(x_a, 0)$ and $\xi_a(x_a, 0)$ and boundary conditions in $d_r(0^-,t)$, $\xi_r(0^-,t)$, and $s_w(0^+,t)$. As discussed in (Jin et al., 2009; Jin, 2010a,b), (3) is also well-defined with some non-invariant flux functions, but interior states different from stationary states can appear at the diverging and merging junctions. Note that the flux functions, (4), (7), and (8), are all invariant, and we can ignore interior states on all links to simplify our analysis.

3 The traffic statics problem

We consider the following constant loading pattern: traffic demand at the origin is always the capacity of link 0, i.e., $d_r(0^-, t) = C_0$; traffic supply at the destination is always the capacity of link 3, i.e., $s_w(0^+, t) = C_3$; and the route choice proportion of commodity 1 vehicles is constant, i.e., $\xi_r(0^-, t) = \xi$. In the traffic statics problem, we are interested in finding stationary solutions for DM2 network with the constant loading pattern.

3.1 Stationary states on a link

In stationary states, total and commodity densities are time-independent on link a ($a = 0, \cdots, 3$):[6]

$$\frac{\partial k_a}{\partial t} = 0, \qquad a = 0, \cdots, 3 \tag{9a}$$

$$\frac{\partial \xi_a k_a}{\partial t} = 0, \qquad a = 0, 3 \tag{9b}$$

From (3a), we have

$$\frac{\mathrm{d}q_a}{\mathrm{d}t} = Q_a'(k_a)\frac{\partial k_a}{\partial t} + \frac{\partial Q_a(k_a)}{\partial x_a}\frac{\mathrm{d}x_a}{\mathrm{d}t} = (Q_a'(k_a) - \frac{\mathrm{d}x_a}{\mathrm{d}t})\frac{\partial k_a}{\partial t} = 0.$$

That is, in stationary states, the total flow-rate on a link is constant:

$$q_a(x_a, t) = q_a, \qquad a = 0, \cdots, 3 \tag{10a}$$

In addition, (3b) yields that $\frac{\partial \xi_a}{\partial t} = \frac{\partial \xi_a}{\partial x_a} = 0$. That is, on link $a = 0, 3$, the proportion of commodity 1 vehicles is the same as the route choice proportion, ξ:

$$\xi_a = \xi. \tag{10b}$$

When link a is stationary with a flow-rate q_a, from (2c) we can see that traffic density $k_a(x_a, t)$ has to be either $k_{a,1} = R_a(q_a/C_a) \le k_{a,c}$ or $k_{a,2} = R_a(C_a/q_a) \ge k_{a,c}$ at any point (a, x_a). From (1), we have $D_a(k_{a,1}) = q_a$, $S_a(k_{a,1}) = C_a$, $D_a(k_{a,2}) = C_a$, and $S_a(k_{a,2}) = q_a$. Thus from the entropy condition (4), there can be the following four possible types of stationary states on link a with flow-rate q_a:

- C (Critical). When $q_a = C_a$, $k_a(x_a, t) = k_{a,1} = k_{a,2} = k_{a,c}$ for $x_a \in [0, X_a]$. That is, link a is stationary at the critical density.
- SUC (Strictly under-critical). When $q_a < C_a$, $k_a(x_a, t)$ can be constant at $k_{a,1}$ for $x_a \in [0, X_a]$. That is, link a is stationary at a strictly under-critical density.

[6] Note that we allow a queue to grow at the origin.

- SOC (Strictly over-critical). When $q_a < C_a$, $k_a(x_a,t)$ can be constant at $k_{a,2}$ for $x_a \in [0, X_a]$. That is, link a is stationary at a strictly over-critical density.
- ZS (Zero shock wave). When $q_a < C_a$, there is a zero shock wave with an upstream density $k_a(x_a,t) = k_{a,1}$ for $x_a \in [0, (1-l_a)X_a)$ and a downstream density $k_a(x_a,t) = k_{a,2}$ for $x_a \in ((1-l_a)X_a, X_a]$, where l_a is the proportion of the congested section. From the LWR theory, it can be verified that there is a shock wave standing at $x_a = (1-l_a)X_a$.

These stationary states are illustrated in Figure 4. Note that, when $q_a < C_a$, at any point x_a, it is not possible to have upstream density $k_a(x_a^-,t) = k_{a,2}$, and downstream density $k_a(x_a^+,t) = k_{a,1}$, since, otherwise, from (4) we have $q_a(x_a,t) = \min\{d_a(x_a^-,t), s_a(x_a^+,t)\} = C_a > q_a$. In a sense, SUC and SOC stationary states can be considered as limiting cases of ZS when $l_a = 0$ and 1, respectively. In addition, C stationary states can be considered as limiting cases of SUC, SOC, and ZS with $q_a = C_a$. With $q_a < C_a$ alone, we are not able to determine the type of stationary states, since l_a can be any number between 0 and 1. In addition, solutions of traffic density with ZS stationary state are not unique, since the shock wave can be at any location on the link.

We denote the demand of the downstream section on link a by d_a and the supply of the upstream section by s_a; i.e.,

$$
\begin{aligned}
d_a(X_a^-,t) &= d_a, && \text{(11a)}\\
s_a(0^+,t) &= s_a. && \text{(11b)}
\end{aligned}
$$

Note that d_a and s_a are defined at different locations. Then we have the following lemma.

Lemma 3.1 *For a stationary state on link a, d_a, s_a, and q_a have to satisfy one of the four conditions: $(d_a,s_a) = (q_a,C_a)$ in SUC ($q_a < C_a$), (C_a,C_a) in C ($q_a = C_a$), (C_a,q_a) in SOC ($q_a < C_a$), and (C_a,C_a) in ZS ($q_a < C_a$).*

If we can determine the pair (d_a,s_a) and q_a in a stationary state, then we can determine the type of a stationary state. If we know the type of a stationary state and q_a, we can uniquely determine (d_a,s_a). Thus, in order to determine the stationary state on link a, we need to find q_a, s_a, and d_a. To uniquely determine a ZS stationary state, we also need to find l_a.

3.2 Stationary entropy conditions at junctions

When all links are in stationary states, the entropy conditions at the origin and destination become

$$
\begin{aligned}
q_r &= q_0 = \min\{C_0, s_0\} = s_0, && \text{(12a)}\\
q_w &= q_3 = \min\{d_3, C_3\} = d_3, && \text{(12b)}
\end{aligned}
$$

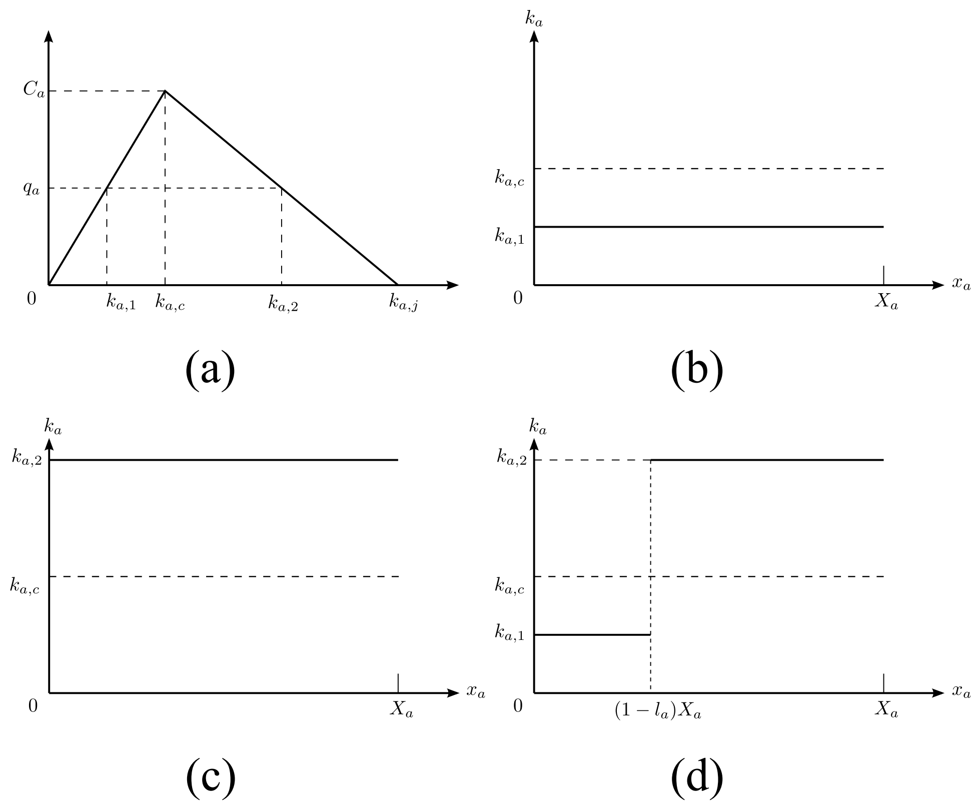

Figure 4: Three types of stationary states on a link when $q_a < C_a$: (a) in the fundamental diagram; (b) an SUC stationary state; (c) an SOC stationary state; and (d) a ZS stationary state

since $s_0 \leq C_0$ and $d_3 \leq C_3$. The entropy condition at the diverge becomes

$$q_0 = \min\{d_0, \frac{s_1}{\xi}, \frac{s_2}{1-\xi}\}, \tag{13a}$$
$$q_1 = q_0 \xi, \tag{13b}$$
$$q_2 = q_0(1-\xi). \tag{13c}$$

The entropy condition at the merge becomes

$$q_3 = \min\{d_1 + d_2, s_3\}, \tag{14a}$$
$$q_1 = \min\{d_1, \max\{s_3 - d_2, \beta s_3\}\}, \tag{14b}$$
$$q_2 = \min\{d_2, \max\{s_3 - d_1, (1-\beta) s_3\}\}. \tag{14c}$$

For the DM2 network, the stationary flow-rates, downstream demands, and upstream supplies of all links have to satisfy (12a), (12b), (13), and (14).

For link 0, since $q_0 = s_0$, from Lemma 3.1 we can see that the stationary state can only be SOC or C; i.e., $d_0 = C_0$; For link 3, since $q_3 = d_3$, from Lemma 3.1 we can see that it can only be SUC or C; i.e., $s_3 = C_3$. In addition, we denote the total network flow-rate by $q = q_3 = q_1 + q_2 = q_0$. Thus the stationary merging and diverging models (13) and (14) can be simplified as

$$q = \min\{C_0, \frac{s_1}{\xi}, \frac{s_2}{1-\xi}\}, \tag{15a}$$
$$q_1 = q\xi, \tag{15b}$$
$$q_2 = q(1-\xi). \tag{15c}$$

and

$$q = \min\{d_1 + d_2, C_3\}, \tag{16a}$$
$$q_1 = \min\{d_1, \max\{C_3 - d_2, \beta C_3\}\}, \tag{16b}$$
$$q_2 = \min\{d_2, \max\{C_3 - d_1, (1-\beta) C_3\}\}. \tag{16c}$$

The two systems of equations lead to

$$q = \min\{d_1 + d_2, C_3\}, \tag{17a}$$
$$\xi q = \min\{d_1, \max\{C_3 - d_2, \beta C_3\}\}, \tag{17b}$$
$$(1-\xi) q = \min\{d_2, \max\{C_3 - d_1, (1-\beta) C_3\}\}, \tag{17c}$$
$$q = \min\{C_0, \frac{s_1}{\xi}, \frac{s_2}{1-\xi}\}. \tag{17d}$$

In the system of equations, (17), given the vector of all links' capacities, $\vec{C} = (C_0, C_1, C_2, C_3)$, the route choice proportion, ξ, and the merging priority, β, there are five unknown variables: q, d_1, d_2, s_1, and s_2. Once we find q, the stationary states on links 0 and 3 can be determined, and the flow-rates on links 1 and 2 are ξq and $(1-\xi)q$, respectively. But note that l_1 and l_2 are not included in (17), and it suggests that the locations of ZS are under-determined.

4 Solutions to the traffic statics problem

When solving the traffic statics problem, network conditions in $\vec{C}$, ξ, and β are assumed to be given. The remaining task is to find from (17) the total flow-rate, q, and stationary states on the two intermediate links, (d_a, s_a) $(a = 1, 2)$. Theoretically, each of links 1 and 2 can be at one of the four types of stationary states: SUC, C, SOC, and ZS. Thus there can be totally 16 combinations of stationary states in the DM2 network. In this section, we first determine the necessary network conditions for each combination of stationary states and then discuss solutions under different network conditions.

4.1 Necessary network conditions for 16 combinations of stationary states

For each of the 16 combinations of stationary states on the DM2 network, we follow the following steps to find its necessary network conditions: First, from Lemma 3.1 we determine the ranges of q and obtain s_1, s_2, d_1, and d_2 in terms of q and ξ; Second, we substitute s_1, s_2, d_1, and d_2 into (17) and obtain four equations involving q, $\vec{C}$, ξ, and β; Third, from the new equations and inequalities we can find q; Finally, we find the necessary conditions of the solution.

As an example, we consider solutions when both links 1 and 2 are stationary at SUC. First, from Lemma 3.1, $q_1 = \xi q < C_1$, and $q_2 = (1-\xi)q < C_2$; i.e.,

$$q < \frac{C_1}{\xi}, \tag{18a}$$

$$q < \frac{C_2}{1-\xi}, \tag{18b}$$

and the demands and supplies can be written in q: $d_1 = q_1 = \xi q$, $d_2 = q_2 = (1-\xi)q$, $s_1 = C_1$, and $s_2 = C_2$. Second, substituting these variables into (17), we obtain

$$\begin{aligned}
q &= \min\{q, C_3\}, && (19a)\\
\xi q &= \min\{\xi q, \max\{C_3 - (1-\xi)q, \beta C_3\}\}, && (19b)\\
(1-\xi)q &= \min\{(1-\xi)q, \max\{C_3 - \xi q, (1-\beta)C_3\}\}, && (19c)\\
q &= \min\{C_0, \frac{C_1}{\xi}, \frac{C_2}{1-\xi}\}. && (19d)
\end{aligned}$$

Third, from (18a), (18b), and (19d), we have $q = C_0$. Finally, from (19a), we have

$$q = C_0 \leq C_3,$$

and (19b) and (19c) are automatically satisfied, since $C_3 - (1-\xi)q \geq \xi q$. Further from (18a) and (18b) we have $C_0 < \frac{C_1}{\xi}$ and $C_0 < \frac{C_2}{1-\xi}$, which lead to

$$1 - \frac{C_2}{C_0} < \xi < \frac{C_1}{C_0}.$$

Link 1	Link 2	q	Existence conditions
SUC	SUC	C_0	$1-\frac{C_2}{C_0}<\xi<\frac{C_1}{C_0}$, $C_0\leq C_3$; $C_0<C_1+C_2$
SUC	C	$C_2/(1-\xi)$	$1-\xi\geq\max\{\frac{C_2}{C_1+C_2}+,\frac{C_2}{C_0},\frac{C_2}{C_3}\}$; $C_2\leq C_0$, $C_2\leq C_3$
SUC	SOC	C_3	$1-\frac{C_2}{C_3}<\xi<\frac{C_1}{C_3}$, $\xi\leq\beta$, $C_3\leq C_0$; $C_3<C_1+C_2$
SUC	ZS	C_3	$1-\frac{C_2}{C_3}<\xi<\frac{C_1}{C_3}$, $\xi\leq\beta$, $C_3=C_0$; $C_0<C_1+C_2$
C	SUC	C_1/ξ	$\xi\geq\max\{\frac{C_1}{C_1+C_2}+,\frac{C_1}{C_0},\frac{C_1}{C_3}\}$; $C_1\leq C_0$, $C_1\leq C_3$
C	C	$C_1/\xi=C_2/(1-\xi)$	$\xi=\frac{C_1}{C_1+C_2}\geq\max\{\frac{C_1}{C_0},\frac{C_1}{C_3}\}$; $C_1+C_2\leq C_0$, $C_1+C_2\leq C_3$
C	SOC	C_3	$1-\frac{C_2}{C_3}<\xi=\frac{C_1}{C_3}$, $\xi\leq\beta$, $C_3\leq C_0$; $C_3<C_1+C_2$
C	ZS	C_3	$1-\frac{C_2}{C_3}<\xi=\frac{C_1}{C_3}$, $\xi\leq\beta$, $C_3\leq C_0$; $C_3<C_1+C_2$
SOC	SUC	C_3	$1-\frac{C_2}{C_3}<\xi<\frac{C_1}{C_3}$, $\xi\geq\beta$, $C_3\leq C_0$; $C_3<C_1+C_2$
SOC	C	C_3	$1-\frac{C_2}{C_3}=\xi<\frac{C_1}{C_3}$, $\xi\geq\beta$, $C_3\leq C_0$; $C_3<C_1+C_2$
SOC	SOC	C_3	$1-\frac{C_2}{C_3}<\xi<\frac{C_1}{C_3}$, $\xi=\beta$, $C_3\leq C_0$; $C_3<C_1+C_2$
SOC	ZS	C_3	$1-\frac{C_2}{C_3}<\xi<\frac{C_1}{C_3}$, $\xi=\beta$, $C_3\leq C_0$; $C_3<C_1+C_2$
ZS	SUC	C_3	$1-\frac{C_2}{C_3}<\xi<\frac{C_1}{C_3}$, $\xi\geq\beta$, $C_3=C_0$; $C_3<C_1+C_2$
ZS	C	C_3	$1-\frac{C_2}{C_3}=\xi<\frac{C_1}{C_3}$, $\xi\geq\beta$ $C_3\leq C_0$; $C_3<C_1+C_2$
ZS	SOC	C_3	$1-\frac{C_2}{C_3}<\xi<\frac{C_1}{C_3}$, $\xi=\beta$, $C_3\leq C_0$; $C_3<C_1+C_2$
ZS	ZS	C_3	$1-\frac{C_2}{C_3}<\xi<\frac{C_1}{C_3}$, $\xi=\beta$, $C_3=C_0$; $C_3<C_1+C_2$

Table 1: Necessary network conditions for 16 combinations of stationary states in the DM2 network: Here $1-\xi\geq\frac{C_2}{C_1+C_2}+$ is equivalent to $1-\xi>\frac{C_2}{C_1+C_2}$.

This condition also implies that $C_0<C_1+C_2$. Therefore, the necessary condition for this combination of stationary states is $1-\frac{C_2}{C_0}<\xi<\frac{C_1}{C_0}$, $C_0\leq C_3$, and $C_0<C_1+C_2$. Note that β is not included in the necessary condition.

In **Table** 1, we show solutions of q and the necessary network conditions for all 16 combinations of stationary states. In this table, each row corresponds to one combination of stationary states on links 1 and 2. Note that $\xi\in[0,1]$ and $\beta\in[0,1]$. For each combination, we can calculate $q_1=q\xi$ and $q_2=q(1-\xi)$, and then determine (d_a,s_a) $(a=1,2)$ from Lemma 3.1.

From **Table** 1, we can see that the total network flow-rate q always solves the following optimization problem:

$$\max q \tag{20}$$

s.t.

$$\begin{aligned} q &\leq C_0, \\ q &\leq C_3, \\ q_1 &= \xi q\leq C_1, \\ q_2 &= (1-\xi)q\leq C_2. \end{aligned}$$

That is, in the DM2 network,

$$q = \min\{C_0, C_3, \frac{C_1}{\xi}, \frac{C_2}{1-\xi}\}. \tag{21}$$

Furthermore, from **Table** 1 we can see that q is strictly smaller than $\min\{C_0, C_1 + C_2, C_3\}$ in SUC-C or C-SUC stationary states. Thus it is possible that the network is under-utilized, since $\min\{C_0, C_1 + C_2, C_3\}$ can be considered as the maximum achievable capacity of the network.

4.2 Stationary states under different network conditions

In **Table** 2, we summarize possible solutions of stationary states under different network conditions. This table is derived from **Table** 1. There are totally 24 sets of network conditions: the first column is for conditions in link capacities, $\vec{C}$, the second column for subsequent conditions in the route choice proportion, ξ, and the third column for further conditions in the merging priority, β. The empty cells in the first two columns have the same conditions as their preceding cells in the same column, but the empty cells in the third column mean that there are no conditions in the merging priority. In the table, the 24 sets of network conditions are mutually exclusive and collectively exhaustive; i.e., they cover all possible DM2 networks in Figure 1. To show this, we first can verify that the four subsets defined by capacities in the first column are mutually exclusive and collectively exhaustive for all possible networks; similarly we can verify that the subsets defined by the route choice proportion and the merging priority are mutually exclusive and collectively exhaustive.

In the fourth column of **Table** 2, all possible combinations of stationary states are listed: slashes separate all possible types of stationary states on the same link, and dashes separate stationary states on links 1 and 2. For example, SUC/SOC/ZS-C means that link 1 can be stationary at SUC, SOC, or ZS, and at the same time link 2 can be stationary at C. The fifth column presents the total flow-rate. For each type of stationary states, $q_1 = q\xi$, $q_2 = q(1-\xi)$, and (d_a, s_a) $(a = 1, 2)$ can be determined from Lemma 3.1. That is, given a set of network conditions in $\vec{C}$, ξ, and β, one can look up **Table** 2 and find possible stationary states, which can be verified to solve (17). Therefore, solutions in **Table** 2 are indeed stationary states for the kinematic wave model in Section 2.2 in the sense that, if the network is initially in a stationary state, then it will stay there.

From **Table** 2, we can see that, given a set of network conditions, stationary flow-rates on all links are unique. However, there can be more than one type of stationary states for eight sets of networks, and multiple stationary states exist if and only if one link can have ZS stationary states. For example, when $C_3 = C_0 < C_1 + C_2$, $\xi = 1 - \frac{C_2}{C_0}$, and $\beta \le \xi$, the total flow-rate is $q = C_3 = C_0$, but there are three types of possible stationary states: SUC-C, SOC-C, or ZS-C. In addition, for a ZS state, the location of the shock wave is under-determined. That is, the total number of vehicles inside the network is under-determined under these network conditions.

Capacities	ξ	β	Link 1-Link 2	q
$C_0 < \min\{C_1 + C_2, C_3\}$	$\xi \leq 1 - \frac{C_2}{C_0}$		SUC-C	$C_2/(1-\xi)$
	$1 - \frac{C_2}{C_0} < \xi < \frac{C_1}{C_0}$		SUC-SUC	C_0
	$\xi \geq \frac{C_1}{C_0}$		C-SUC	C_1/ξ
$\min\{C_0, C_3\} \geq C_1 + C_2$	$\xi < \frac{C_1}{C_1+C_2}$		SUC-C	$C_2/(1-\xi)$
	$\xi = \frac{C_1}{C_1+C_2}$		C-C	C_1/ξ
	$\xi > \frac{C_1}{C_1+C_2}$		C-SUC	C_1/ξ
$C_3 = C_0 < C_1 + C_2$	$\xi < 1 - \frac{C_2}{C_0}$		SUC-C	$C_2/(1-\xi)$
	$\xi = 1 - \frac{C_2}{C_0}$	$\beta > \xi$	SUC-C	C_3
		$\beta \leq \xi$	SUC/SOC/ZS-C	C_3
	$1 - \frac{C_2}{C_0} < \xi < \frac{C_1}{C_0}$	$\beta > \xi$	SUC-SUC/SOC/ZS	C_3
		$\beta < \xi$	SUC/SOC/ZS-SUC	C_3
		$\beta = \xi$	SUC/SOC/ZS-SUC/SOC/ZS	C_3
	$\xi = \frac{C_1}{C_0}$	$\beta < \xi$	C-SUC	C_3
		$\beta \geq \xi$	C-SUC/SOC/ZS	C_3
	$\xi > \frac{C_1}{C_0}$		C-SUC	C_1/ξ
$C_3 < \min\{C_0, C_1 + C_2\}$	$\xi < 1 - \frac{C_2}{C_3}$		SUC-C	$C_2/(1-\xi)$
	$\xi = 1 - \frac{C_2}{C_3}$	$\beta > \xi$	SUC-C	C_3
		$\beta \leq \xi$	SUC/SOC/ZS-C	C_3
	$1 - \frac{C_2}{C_3} < \xi < \frac{C_1}{C_3}$	$\beta > \xi$	SUC-SOC	C_3
		$\beta < \xi$	SOC-SUC	C_3
		$\beta = \xi$	SOC-SUC/SOC/ZS, SUC/ZS-SOC	C_3
	$\xi = \frac{C_1}{C_3}$	$\beta < \xi$	C-SUC	C_3
		$\beta \geq \xi$	C-SUC/SOC/ZS	C_3
	$\xi > \frac{C_1}{C_3}$		C-SUC	C_1/ξ

Table 2: Possible stationary states under different network conditions: In the fourth column, the stationary states on the left of a dash are for link 1, and those on the right for link 2. A slash means "or".

5 Two special merging rules

In this section, we consider two special merging priorities: $\beta = \frac{C_1}{C_1+C_2}$ and $\beta = 0$. In the first case, vehicles on the two intermediate links merge in a fair fashion according to link capacity ratios; in the second case, link 2 has the absolute merging priorities. These two merging priorities represent two extreme cases of (8).

5.1 Stationary states with a fair merging rule

When $\beta = \frac{C_1}{C_1+C_2}$, the merge model (8) is equivalent to the fair merge model proposed in (Jin and Zhang, 2003; Ni and Leonard, 2005; Jin, 2010b), in which the distribution of the total flow-rate to an upstream link is proportional to its demand. However, it was shown that the DM2 network cannot reach stationary states when it is initially empty under certain network conditions. For example, when $\vec{C} = (3, 1, 2, 2)$ and $\xi = 0.45$, persistent oscillations occur in the DM2 network. This contradicts to that stationary states exist under all network conditions. In this subsection we attempt to resolve the discrepancy.

In **Table** 3, we summarize all stationary solutions given by Corollary 3.2 in (Jin, 2009). Note that the cases when $C_0 < \min\{C_1 + C_2, C_3\}$ and $C_3 = C_0 < C_1 + C_2$ are considered separately in **Table** 3, and UC states in Corollary 3.2 of (Jin, 2009) are replaced by SUC states. From the table we can see that only seven combinations of stationary states on links 1 and 2 can be reached in an initially empty DM2 network during a finite period of time: SUC/C-SUC/C, SOC-SOC/Z, or ZS-SOC. The other nine combinations, SOC/ZS-SUC/C, SUC/C-SOC/ZS, or ZS-ZS, cannot be reached. In particular, when $C_3 < \min\{C_0, C_1 + C_2\}$ and $1 - \frac{C_2}{C_3} < \xi < \frac{C_1}{C_1+C_2}$ or $\frac{C_1}{C_1+C_2} < \xi < \frac{C_1}{C_3}$, an initially empty DM2 network cannot reach any stationary states during a finite period of time; instead, damped periodic oscillations (DPO) or persistent periodic oscillations (PPO) will occur under these network conditions.

When $\beta = \frac{C_1}{C_1+C_2}$, the necessary conditions for C-SOC, C-ZS, SOC-C, or ZS-C stationary states in **Table** 1 cannot be satisfied. That is, when one link is stationary at C, the other has to be at C or SUC. Therefore, there can exist only 12 types of stationary states with the fair merging rule, and their necessary conditions are listed in **Table** 4. Furthermore, stationary solutions under different network conditions are given in **Table** 5.

Comparing **Table** 3 and **Table** 5, we have the following observations:

1. When $C_3 = C_0 < C_1 + C_2$ and $\xi \in (1 - \frac{C_2}{C_0}, \frac{C_1}{C_0})$, we find multiple stationary states, but only SUC-SUC states are possible in initially empty networks.

2. When $C_3 < \min\{C_0, C_1 + C_2\}$ and $\xi = \frac{C_1}{C_1+C_2}$, we find totally five types of stationary states, but only three types are possible in initially empty networks, without SUC-SOC or SOC-SUC solutions.

3. When $C_3 < \min\{C_0, C_1 + C_2\}$ and $\xi \in (1 - \frac{C_2}{C_3}, \frac{C_1}{C_1+C_2}) \cup (\frac{C_1}{C_1+C_2}, \frac{C_1}{C_3})$, we find SUC-SOC or SOC-SUC stationary solutions, but no stationary states were found, and DPO or PPO solutions occur in initially empty networks.

Capacities	ξ	Link 1-Link 2	q
$C_0 < \min\{C_1+C_2, C_3\}$	$\xi \leq 1-\frac{C_2}{C_0}$	SUC-C	$C_2/(1-\xi)$
	$1-\frac{C_2}{C_0} < \xi < \frac{C_1}{C_0}$	SUC-SUC	C_0
	$\xi \geq \frac{C_1}{C_0}$	C-SUC	C_1/ξ
$\min\{C_0, C_3\} \geq C_1+C_2$	$\xi < \frac{C_1}{C_1+C_2}$	SUC-C	$C_2/(1-\xi)$
	$\xi = \frac{C_1}{C_1+C_2}$	C-C	C_1/ξ
	$\xi > \frac{C_1}{C_1+C_2}$	C-SUC	C_1/ξ
$C_3 = C_0 < C_1+C_2$	$\xi \leq 1-\frac{C_2}{C_0}$	SUC-C	$C_2/(1-\xi)$
	$1-\frac{C_2}{C_0} < \xi < \frac{C_1}{C_0}$	SUC-SUC	C_3
	$\xi \geq \frac{C_1}{C_0}$	C-SUC	C_1/ξ
$C_3 < \min\{C_0, C_1+C_2\}$	$\xi \leq 1-\frac{C_2}{C_3}$	SUC-C	$C_2/(1-\xi)$
	$1-\frac{C_2}{C_3} < \xi < \frac{C_1}{C_1+C_2}$	DPO or PPO	-
	$\xi = \frac{C_1}{C_1+C_2}$	SOC-SOC/ZS or ZS-SOC	C_3
	$\frac{C_1}{C_1+C_2} < \xi < \frac{C_1}{C_3}$	DPO or PPO	-
	$\xi \geq \frac{C_1}{C_3}$	C-SUC	C_1/ξ

Table 3: Stationary states that can be reached in a finite period of time in an initially empty DM2 network with the fair merging rule: In the third column, the stationary states on the left of a dash are for link 1, those on the right for link 2, and a slash means ‘‘or’’. Results are from Corollary 3.2 in (Jin, 2009).

Link 1	Link 2	q	Existence conditions
SUC	SUC	C_0	$1-\frac{C_2}{C_0} < \xi < \frac{C_1}{C_0}$, $C_0 \leq C_3$; $C_0 < C_1+C_2$
SUC	C	$C_2/(1-\xi)$	$1-\xi \geq \max\{\frac{C_2}{C_1+C_2}+, \frac{C_2}{C_0}, \frac{C_2}{C_3}\}$; $C_2 \leq C_0$, $C_2 \leq C_3$
SUC	SOC	C_3	$1-\frac{C_2}{C_3} < \xi \leq \frac{C_1}{C_1+C_2}$, $C_3 \leq C_0$; $C_3 < C_1+C_2$
SUC	ZS	C_3	$1-\frac{C_2}{C_3} < \xi \leq \frac{C_1}{C_1+C_2}$, $C_3 = C_0$; $C_0 < C_1+C_2$
C	SUC	C_1/ξ	$\xi \geq \max\{\frac{C_1}{C_1+C_2}+, \frac{C_1}{C_0}, \frac{C_1}{C_3}\}$; $C_1 \leq C_0$, $C_1 \leq C_3$
C	C	$C_1/\xi = C_2/(1-\xi)$	$\xi = \frac{C_1}{C_1+C_2} \geq \max\{\frac{C_1}{C_0}, \frac{C_1}{C_3}\}$; $C_1+C_2 \leq C_0$, $C_1+C_2 \leq C_3$
SOC	SUC	C_3	$\frac{C_1}{C_1+C_2} \leq \xi < \frac{C_1}{C_3}$, $C_3 \leq C_0$; $C_3 < C_1+C_2$
SOC	SOC	C_3	$\xi = \frac{C_1}{C_1+C_2}$, $C_3 \leq C_0$; $C_3 < C_1+C_2$
SOC	ZS	C_3	$\xi = \frac{C_1}{C_1+C_2}$, $C_3 \leq C_0$; $C_3 < C_1+C_2$
ZS	SUC	C_3	$\frac{C_1}{C_1+C_2} \leq \xi < \frac{C_1}{C_3}$, $C_3 = C_0$; $C_3 < C_1+C_2$
ZS	SOC	C_3	$\xi = \frac{C_1}{C_1+C_2}$, $C_3 \leq C_0$; $C_3 < C_1+C_2$
ZS	ZS	C_3	$\xi = \frac{C_1}{C_1+C_2}$, $C_3 = C_0$; $C_3 < C_1+C_2$

Table 4: Necessary conditions for 12 combinations of stationary states on links 1 and 2 with the fair merging rule: Here $1-\xi \geq \frac{C_2}{C_1+C_2}+$ is equivalent to $1-\xi > \frac{C_2}{C_1+C_2}$.

Capacities	ξ	Link 1-Link 2	q
$C_0 < \min\{C_1+C_2, C_3\}$	$\xi \leq 1-\frac{C_2}{C_0}$	SUC-C	$C_2/(1-\xi)$
	$1-\frac{C_2}{C_0} < \xi < \frac{C_1}{C_0}$	SUC-SUC	C_0
	$\xi \geq \frac{C_1}{C_0}$	C-SUC	C_1/ξ
$\min\{C_0, C_3\} \geq C_1+C_2$	$\xi < \frac{C_1}{C_1+C_2}$	SUC-C	$C_2/(1-\xi)$
	$\xi = \frac{C_1}{C_1+C_2}$	C-C	C_1/ξ
	$\xi > \frac{C_1}{C_1+C_2}$	C-SUC	C_1/ξ
$C_3 = C_0 < C_1+C_2$	$\xi \leq 1-\frac{C_2}{C_0}$	SUC-C	$C_2/(1-\xi)$
	$1-\frac{C_2}{C_0} < \xi < \frac{C_1}{C_1+C_2}$	SUC-SUC/SOC/ZS	C_3
	$\xi = \frac{C_1}{C_1+C_2}$	SUC/SOC/ZS-SUC/SOC/ZS	C_3
	$\frac{C_1}{C_1+C_2} < \xi < \frac{C_1}{C_0}$	SUC/SOC/ZS-SUC	C_3
	$\xi \geq \frac{C_1}{C_0}$	C-SUC	C_1/ξ
$C_3 < \min\{C_0, C_1+C_2\}$	$\xi \leq 1-\frac{C_2}{C_3}$	SUC-C	$C_2/(1-\xi)$
	$1-\frac{C_2}{C_3} < \xi < \frac{C_1}{C_1+C_2}$	SUC-SOC	C_3
	$\xi = \frac{C_1}{C_1+C_2}$	SOC-SUC/SOC/ZS, SUC/ZS-SOC	C_3
	$\frac{C_1}{C_1+C_2} < \xi < \frac{C_1}{C_3}$	SOC-SUC	C_3
	$\xi \geq \frac{C_1}{C_3}$	C-SUC	C_1/ξ

Table 5: Solutions of stationary states with the fair merging rule: In the third column, the stationary states on the left of a dash are for link 1, those on the right for link 2, and a slash means “or”.

Thus we can conclude that stationary solutions to the traffic statics problem always exist, but they may not be reached in an initially empty DM2 network. In particular, when $C_3 < \min\{C_0, C_1 + C_2\}$ and $\xi \in (1 - \frac{C_2}{C_3}, \frac{C_1}{C_1+C_2})$ or $\xi \in (\frac{C_1}{C_1+C_2}, \frac{C_1}{C_3})$, the DM2 network can have SUC-SOC or SOC-SUC stationary states, but they cannot be reached in a finite period of time in initially empty DM2 networks. This suggests that some stationary states may be unstable, and the stability of stationary states will be subject to discussions in a follow-up study.

5.2 Stationary states with an absolutely prioritized merging rule

When $\beta = 0$, link 2 has the absolute local merging priority; i.e., traffic demand from link 2 is always satisfied before vehicles from link 1 are allowed to enter link 3. In this case, $\xi \leq \beta$ is equivalent to $\xi = 0$. Under different network conditions, possible solutions of stationary states are listed in **Table** 6, from which we can see that stationary states always exist. In particular, when $\xi = 0$, $q_1 = 0$, and link 1 can still be SOC when it is totally jammed. From **Table** 6, we can see that the total flow-rate q also satisfies (21), which is independent of β.

Note that $\beta = 0$ is different from a global priority-based merging rule discussed in the Conclusion section of (Jin, 2009), where vehicles on link 1 have to wait until link 2 is empty. It can be shown that with the global priority-based merging rule, q_1 has to be 0 in stationary states; i.e., stationary states only exist when $\xi = 0$. This does not contradict our results, since our merging model in (8) is always local.

6 Conclusion

Within the framework of multi-commodity kinematic wave models, we defined and solved the traffic statics problem in an open diverge-merge network under constant boundary and network conditions. We discussed the properties of four types of stationary states on a link and derived stationary entropy conditions at network junctions. We proved that, in the DM2 network, stationary solutions always exist under different network conditions. But when one link carries a zero-speed shock wave, the network can have multiple stationary states: across different solutions, flow-rates on all links are the same, but traffic patterns and, therefore, traffic densities and travel times can be significantly different. This suggests that the flow-density relations in macroscopic fundamental diagrams are not unique on a road network, as observed in (Daganzo et al., 2011).

In this study, the traffic statics problem is defined within the framework of network kinematic wave models, for which invariant junction flux functions are used as entropy conditions at the diverge and the merge. It is important to note that non-invariant junction flux functions can also be used, but the analysis would be much more cumbersome. We expect that, for general networks, invariant junction flux functions are even more critical for solving the traffic statics problem. In addition, it is also possible to define the network traffic

Capacities	ξ	$\beta = 0$	Link 1-Link 2	q
$C_0 < \min\{C_1 + C_2, C_3\}$	$\xi \leq 1 - \frac{C_2}{C_0}$		SUC-C	$C_2/(1-\xi)$
	$1 - \frac{C_2}{C_0} < \xi < \frac{C_1}{C_0}$		SUC-SUC	C_0
	$\xi \geq \frac{C_1}{C_0}$		C-SUC	C_1/ξ
$\min\{C_0, C_3\} \geq C_1 + C_2$	$\xi < \frac{C_1}{C_1+C_2}$		SUC-C	$C_2/(1-\xi)$
	$\xi = \frac{C_1}{C_1+C_2}$		C-C	C_1/ξ
	$\xi > \frac{C_1}{C_1+C_2}$		C-SUC	C_1/ξ
$C_3 = C_0 < C_1 + C_2$	$\xi < 1 - \frac{C_2}{C_0}$		SUC-C	$C_2/(1-\xi)$
	$\xi = 1 - \frac{C_2}{C_0}$	$0 \leq \xi$	SUC/SOC/ZS-C	C_3
	$1 - \frac{C_2}{C_0} < \xi < \frac{C_1}{C_0}$	$0 < \xi$	SUC/SOC/ZS-SUC	C_3
		$0 = \xi$	SUC/SOC/ZS-SUC/SOC/ZS	C_3
	$\xi = \frac{C_1}{C_0}$	$0 < \xi$	C-SUC	C_3
	$\xi > \frac{C_1}{C_0}$		C-SUC	C_1/ξ
$C_3 < \min\{C_0, C_1 + C_2\}$	$\xi < 1 - \frac{C_2}{C_3}$		SUC-C	$C_2/(1-\xi)$
	$\xi = 1 - \frac{C_2}{C_3}$	$0 \leq \xi$	SUC/SOC/ZS-C	C_3
	$1 - \frac{C_2}{C_0} < \xi < \frac{C_1}{C_0}$	$0 < \xi$	SOC-SUC	C_3
		$0 = \xi$	SOC-SUC/SOC/ZS, SUC/ZS-SOC	C_3
	$\xi = \frac{C_1}{C_3}$	$0 < \xi$	C-SUC	C_3
	$\xi > \frac{C_1}{C_3}$		C-SUC	C_1/ξ

Table 6: Stationary states with $\beta = 0$: In the fourth column, the stationary states on the left of a dash are for link 1, and those on the right for link 2. A slash means ‘‘or’’.

statics problem with other types of traffic flow models, such as with microscopic car-following and lane-changing models, but its solutions may not exist in a road network.

By comparing results with those in (Jin, 2009), we found that stationary states exist even under network conditions when an initially empty DM2 network can settle in persistent periodic oscillations after a long time. This suggests that the occurrence of specific stationary states would depend on the initial conditions, and certain stationary states may not be stable in the DM2 network. In a follow-up study, we will examine the stability property of stationary states in the DM2 network. In the future, we will also be interested in studying the traffic statics problem in general networks, which can be either open, closed, or mixed. Such studies would be helpful to further reveal the impacts of network topologies on traffic dynamics.

In this study, we developed a brute-force method by enumerating all 16 combinations of stationary states in the DM2 network. The method can provide us a complete picture of stationary states in small road networks: for example, it can also be applied to analyze all possible stationary states in a closed double-ring network (Daganzo et al., 2011; Gayah and Daganzo, 2011). However, this method is not efficient for more general networks: if each link can have four types of stationary states, then a network with $|A|$ links will have $4^{|A|}$ combinations of stationary states. Even for the DM2 network, if the origin demand and destination supply are $d_r(0^-,t) = d_r < C_0$ and $s_w(0^+,t) = s_w < C_3$, each of links 0 and 3 can be at four types of stationary states, and the whole DM2 network can have $4^4 = 256$ combinations of stationary states under the general demand patterns. [7] Therefore, for general road networks, more efficient methods have to be developed to solve the traffic statics problem.

In the future, we will be interested in studying various traffic problems in stationary road networks. For example, one can study the macroscopic fundamental diagram in a stationary road network (Daganzo and Geroliminis, 2008; Geroliminis and Daganzo, 2008). In addition, one can study the design, management, and control of road networks in stationary states. For examples, in a freeway network shown in Figure 2 or a grid network shown in Figure 3, we can design the metering algorithm or the signal timing scheme to avoid oscillations in a road network and achieve the maximum throughput; we can solve the dynamic traffic assignment problem in stationary states discussed in (Merchant and Nemhauser, 1978), but the stationary states are defined within the framework of network kinematic wave models, which are able to account for interactions among traffic streams and queue spill-backs. In stationary road networks, these problems could be mathematically tractable, and their solutions can provide more insights into more realistic problems with time-dependent demand patterns and dynamic traffic conditions.

[7] We expect that the network throughput becomes $q = \min\{d_r, C_0, \frac{C_1}{\xi}, \frac{C_2}{1-\xi}, C_3, s_w\}$, but it will be cumbersome at the least to enumerate all 256 types of stationary states.

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